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Real Analysis

Sequence - A function u whose domain is the set of natural numbers N is called a sequence. Thus a real sequence u is a correspondence which associates each natural number to uniquely determined real number. The image of the natural number n is denoted by u_n or $u(n)$.

A sequence u is denoted by $u_1, u_2, u_3, ---, u_n, ---$ in the long hand and briefly it is written as $\{u_n\}$ or $< u_n >$ or u_n.

The image of n i.e. u_n is called the nth term or nth element of the sequence

Ex. ① $1, 3, 5, 7, ---, 2n-1, ---$ is a sequence for which $u_1 = 1$, $u_2 = 3$, $---$, $u_n = 2n-1$, etc

② $-1, +1, -1, +1, ---$ is a sequence for which $u_1 = -1$, $u_2 = +1 = (-1)^2$, $---$, $u_n = (-1)^n$, etc

The set $\{x \mid x = u_n, n \in N\}$, of distinct terms of the sequence $\{u_n\}$, is called the range or range-set of the sequence $\{u_n\}$. The set of terms of the sequence $\{u_n\}$ is the set $u = \{u_1, u_2, u_3, ---, u_n, ---\}$.

Ex. The range set of $\{u_n\}$, $u_n = 3n+1$ is the set $\{4, 7, 10, 13, \cdots\}$ which is subset of R.

From the definition of sequence as a function from N to R, it is clear that the number of terms (distinct or equal) in a sequence $\{u_n\}$ is infinite.

(i) $1, 3, 5, 7, ---, 25$; the sequence is $\{2n-1\}$

(ii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ----, \frac{1}{2^{n-1}}$; the sequence is $\left\{\frac{1}{2^{n-1}}\right\}$.

Bounded Sequences – The sequence $\{u_n\}$ is said to be bounded above if there exists a finite number $K \in R$ such that $u_n \leq K$, for all values of $n \in N$.

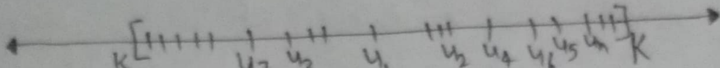
The sequence $\{u_n\}$ is said to be bounded below if there exists a finite number $k \in R$ such that $u_n \geq k$, for all values of $n \in N$.

The sequence $\{u_n\}$, which is bounded above and below is said to be bounded, and the numbers K and k are called rough upper and lower bounds respectively.

$\therefore$ u_n is a bounded sequence if $k \leq u_n \leq K \ \forall\ n \in N$.

In other words $\{u_n\}$ is bounded if $u_n \in [k, K] \ \forall\ n \in N$.

Pictorially



Least upper and Greatest lower Bounds

If M is a number such that

(i) $u_n \leq M$, for all values of $n \in N$.

and (ii) $u_n > M - \epsilon$, for at least one value of $n \in N$ where ϵ is an arbitrary positive number, however small, then M is called the least upper bound or simply the upper bound of the sequence $\{u_n\}$, which is bounded above.

Clearly the upper bound of $\{u_n\}$ is also a rough upper bound of $\{u_n\}$.

If m is a number such that

(i) $u_n \geq m$ for all values of n

and (ii) $u_n < m + \epsilon$, for at least one value of $n \in N$ where ϵ is an arbitrary positive number, however small, then m is called the greatest lower bound or simply the lower bound of the sequence $\{u_n\}$, which is bounded below.

Clearly the lower bound of $\{u_n\}$ is also a rough lower bound of $\{u_n\}$.

Ex. (i) The sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{n+1}{n}, \ldots$ is bounded, as its upper bound is 2 and lower bound is 1. A rough upper bound is 3 and a rough ~~rough~~ lower bound is 0.

(ii) The sequence $1, 2, 3, \ldots, n, \ldots$ is bounded below but it is not bounded above and hence it is not bounded.